

Semiclassical string spectrum in a string model dual to large N QCD

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We explore the string spectrum in the Witten QCD₄ model by considering classical string configurations, thereby obtaining energy formulas for quantum states with large excitation quantum numbers representing glueballs and Kaluza-Klein states. In units of the string tension, the energies of all states increase as the 't Hooft coupling λ is decreased, except the energies of glueballs corresponding to strings lying on the horizon, which remain constant. We argue that some string solutions can be extrapolated to the small λ regime. We also find the classical mechanics description of supergravity glueballs in terms of point-like string configurations oscillating in the radial direction, and reproduce the glueball energy formula previously obtained by solving the equation for the dilaton fluctuation.

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1. Introduction

Using the AdS/CFT correspondence [1], Witten proposed a string model representing a holographic dual to pure $SU(N)$ 3 + 1 Yang-Mills theory [2]. The idea is to start with N D4 branes compactified on a circle of radius r_0 , and to impose anti-periodic boundary conditions for the fermions. Then fermions get masses of order r_0^{-1} and scalar particles get masses at one loop. The resulting low energy theory on the D4 brane is then pure $SU(N)$ Yang-Mills theory in 3+1 dimensions, describing the dynamics of massless gluons. The corresponding supergravity background can be constructed in terms of the near-horizon limit of the euclidean non-extremal D4 brane. This prescribes the correct anti-periodic boundary conditions for the fermions.

The string spectrum in this supergravity background contains particles with Kaluza-Klein S^4 quantum numbers. These should decouple, since they have no counterpart in QCD, which does not have $SO(5)$ global symmetry. The masses of these particles have been computed in the supergravity sector [3,4,5], and, in the supergravity approximation, they are basically of the same order of glueball masses. In addition, there are Kaluza-Klein particles with S^1 quantum numbers which should also decouple, in order for the string model to describe 3+1 dimensional rather than 4+1 dimensional Yang-Mills theory. They also have masses of the same order as glueball masses. Such states can be decoupled by adding rotation parameters to the supergravity background, which breaks the non abelian $SO(5)$ global symmetries to the Cartan subgroup $U(1) \times U(1)$ [6,7,8,9]. But even in this case, the S^4 Kaluza-Klein modes do not decouple within the supergravity approximation.

Nevertheless, the model reproduces in a quite remarkable way several qualitative and quantitative features of QCD. It describes the correct N dependence of physical quantities like e.g. a hadron spectrum independent of N [2] and a gluon condensate proportional to N^2 [10]. It also exhibits confinement in the form of an area law for Wilson loops [2]. Confinement in this model was also shown to be accompanied by monopole condensation [11], as expected.

Moreover, by adding N_f D6 brane probes to this D4 brane supergravity background, Kruczenski, Mateos, Myers and Winters [12] recently constructed a string model dual to four dimensional QCD with N colours and N_f flavours of mass m_q . They have shown that for $m_q = 0$ the model exhibits spontaneous chiral symmetry breaking by a quark condensate, extending previous results on chiral symmetry breaking condensates found in a different gravity dual model in [13]. For $m_q > 0$ the pion mass exactly satisfies the Gell-Mann-Oakes-Renner formula which relates it to the quark condensate. In addition, the model provides a simple geometric interpretation of the Vafa-Witten theorem [12], and the authors of [14] found evidence that the Veneziano-Witten theorem (relating the mass of the η' to the topological susceptibility computed in [10]) is satisfied (the Veneziano-Witten formula has been derived in a setup of D3 branes on orbifold singularities in [15]).

The model was also used to describe the recently observed pentaquarks [16], obtaining reasonable predictions for masses.

Computing the full string spectrum in this string model of large N QCD may give important insights on general features of the glueball spectrum of large N QCD and on what are the scales associated with the Kaluza-Klein particles of the model. The determination of the string spectrum in a curved background is in general a complicated problem. However, as suggested in [17], one can explore the spectrum at large quantum numbers by considering classical string configurations as well as quantum fluctuations about it. This approach led to remarkable results in the $AdS_5 \times S^5$ case (see e.g. [18] and references therein). In this paper we will consider a large variety of classical string configurations with the aim of determining the typical energy scales at strong λ coupling of physical string states with different quantum numbers.¹ We will find a universal pattern, indicating that for any string solution the derivative of the ratio $\text{Energy}/\sqrt{T_{\text{YM}}}$ with respect to λ (T_{YM} is the string tension) is negative or vanishing, but never positive. In other words, to leading order at $\lambda \gg 1$, the energies of the states in units of the string tension increase or remain constant as λ is decreased. Some states become unstable when the angular momentum is lower than some critical value. Remarkably, this occurs precisely at the point where the derivative of $\frac{E}{\sqrt{T_{\text{YM}}}}$ with respect to λ becomes positive, thus confirming the rule that, in units of the string tension, the energies of the states increase or remain constant in decreasing λ (in the region $\lambda \gg 1$). We also find some string solutions which exist only for special values of the coupling λ .

2. QCD₃₊₁ model: generalities and restrictions

The background of the Witten QCD model can be obtained from the euclidean non-extremal D4 brane (we follow the notation of [6]),

$$ds_{10}^2 = \alpha' \left(\frac{8\pi\lambda}{3} \frac{u^3}{u_0} (dx_1^2 + \dots + dx_4^2) + \frac{8\pi\lambda}{27} \left(\frac{u}{u_0} \right)^3 h(u) d\theta_1^2 + \frac{8\pi\lambda}{3} \frac{du^2}{u u_0 h(u)} + \frac{2\pi\lambda}{3} \frac{u}{u_0} d\Omega_4^2 \right), \quad (2.1)$$

$$e^{2\phi} = \frac{8\pi}{27} \frac{\lambda^3 u^3}{u_0^3} \frac{1}{N^2}, \quad h(u) \equiv 1 - \frac{u_0^6}{u^6},$$

$$d\Omega_4^2 = d\alpha^2 + \cos^2 \alpha (d\theta^2 + \cos^2 \theta d\varphi^2 + \sin^2 \theta d\varphi^2), \quad (2.2)$$

where λ is the 't Hooft coupling, $2\pi\lambda = g_{\text{YM}}^2 N$. The Euclidean time is $t_E = R_0 \theta_1$, $R_0 = (3u_0)^{-1}$, and θ_1 is 2π periodic. To describe the Minkowskian QCD₃₊₁ theory, one

¹ Previous discussions on semiclassical string solutions in this model are in [19,20]. Studies of semiclassical string solutions in other confining backgrounds can be found in [21].

makes the Wick rotation $x_4 \rightarrow ix_0$. The metric exhibits a coordinate singularity at $u = u_0$, which can be removed as usual by going to e.g. Rindler-type coordinates

$$\frac{u^3}{u_0^3} = 1 + \frac{\rho^2}{\rho_0^2} \equiv h_1(\rho) , \quad \rho_0^2 \equiv u_0^3 , \quad \rho \in [0, \infty) .$$

One finds

$$\begin{aligned} ds_{10}^2 = \alpha' \left(\frac{8\pi\lambda u_0^2}{3} h_1(-dx_0^2 + dx_1^2 + \dots + dx_3^2) + \frac{16\pi\lambda}{27} \frac{h_2}{h_1} \frac{\rho^2}{\rho_0^2} d\theta_1^2 \right. \\ \left. + \frac{16\pi\lambda}{27\rho_0^2} \frac{h_1^{1/3}}{h_2} d\rho^2 + \frac{2\pi\lambda}{3} h_1^{1/3} d\Omega_4^2 \right) . \end{aligned} \quad (2.3)$$

$$e^{2\phi} = \frac{8\pi}{27} \lambda^3 h_1 \frac{1}{N^2} , \quad h_2 \equiv 1 + \frac{\rho^2}{2\rho_0^2} .$$

In this model, QCD arises as the dimensional reduction of five dimensional Yang-Mills theory, which is compactified on a circle of radius $(3u_0)^{-1}$ with anti-periodic boundary conditions for the fermions. From the point of view of the gauge theory, the parameter u_0 plays the role of an ultraviolet cutoff, and the coupling g_{YM} should be understood as the bare coupling at distances corresponding to $1/u_0$. There are Kaluza-Klein particles with masses of order $O(u_0)$. To obtain $d = 3 + 1$ QCD, one should take $u_0 \rightarrow \infty$ and $\lambda \rightarrow 0$ with a fixed string tension [2,11], which for small λ should be of the form $T_{\text{YM}} \sim \exp(-2b/\lambda) u_0^2 \sim \Lambda_{\text{QCD}}^2$.

In QCD, the glueball mass scale is Λ_{QCD} , so it is essentially set by the square root of the string tension. In the present supergravity background, there are glueball states whose masses are proportional to u_0 for $\lambda \gg 1$, with a coefficient independent of λ , which originate from the supergravity sector of the string spectrum, and, as we shall see, glueball states of masses proportional to the square root of the string tension, which at $\lambda \gg 1$ is given by

$$T_{\text{YM}} = \frac{4}{3} \lambda u_0^2 , \quad \lambda \gg 1 . \quad (2.4)$$

This is the case for highly excited string states, as in the example shown in the next section. This tension derived from the energy formula (3.8) agrees with the tension obtained from the Wilson loop calculation (see [2,22,11]). Thus the masses of these two sets of states differ by a factor $\sqrt{\lambda}$. Since in pure gluodynamics there is only a single mass scale, one expects that the states surviving the limit $\lambda \ll 1$ will all have masses of the same order, proportional to the square root of the string tension.

To take the small λ limit in the geometry, one defines $u' = \lambda u / u_0$ so that the dilaton coupling (2.1) is fixed in taking the limit $\lambda \rightarrow 0$ at fixed u' . This gives the extremal D4 brane, which is singular at $u' = 0$. In this region, the supergravity approximation cannot

be justified. The gravity solution (2.1) solves the leading α' order string equations, and therefore it applies as long as further α' corrections are suppressed. To ensure that these corrections are small one is to demand that curvature invariants and higher derivatives of the dilaton are small in string units. In the present case, these are all suppressed by inverse powers of λ . In particular, the curvature scalar is $\alpha'\mathcal{R} = -\frac{27}{8\pi\lambda}\frac{u_0}{u}(5 - \frac{u_0^6}{u^6})$, which is small in all the space provided $\lambda \gg 1$.

In addition to the requirement of large λ , one must require that the string coupling constant is small, so that string loop corrections are suppressed. This implies the condition $u < u_{max}$, with $u_{max} \approx \frac{N^{\frac{2}{3}}}{\lambda}u_0$. By taking N sufficiently large, one can extend the applicability of the tree-level approximation to arbitrary large values of u .

3. Semiclassical string spectrum

In the physical string spectrum, there are basically two types of string states:

- 1) The states having zero quantum numbers in S^1 and S^4 , which in the small λ limit should correspond to glueballs of QCD.
- 2) The ‘‘Kaluza-Klein’’ type of string states, having non-zero quantum numbers in S^1 or in S^4 . Such states should decouple in the small $\lambda \ll 1$ limit since they have no counterpart in ordinary QCD.

In this section we compare the masses of semiclassical string states describing glueball states and Kaluza-Klein states. We will argue that they correspond to two different mass scales which get separated as λ is decreased, giving rise to heavy Kaluza-Klein states and lighter glueball states.

The states are characterized by angular momentum quantum numbers, which are conserved, and by winding numbers of the strings around S^1 and S^4 —or number of foldings of a folded string—which are not conserved in interactions, because all circles are contractible. Nevertheless, they can be used to characterize the string states of the free string theory.

In order to obtain the relations between the energy and the angular momenta of classical string configurations, one can start with the Polyakov action in the conformal gauge,

$$I = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma G_{\mu\nu} \partial_\alpha X^\mu(\tau, \sigma) \partial_\beta X^\nu(\tau, \sigma) \eta^{\alpha\beta}. \quad (3.1)$$

The classical strings must satisfy the equations of motion

$$\left(-\frac{\partial G_{\rho\nu}}{\partial X^\mu} + 2\frac{\partial G_{\mu\nu}}{\partial X^\rho}\right) \left(\dot{X}^\rho \dot{X}^\nu - X^{\rho'} X^{\nu'}\right) + 2G_{\mu\nu} \left(\ddot{X}^\nu - X^{\nu''}\right) = 0, \quad (3.2)$$

where dots and primes denote derivatives with respect to τ and σ , respectively. In addition, one has the usual Virasoro constraints

$$G_{\mu\nu}(\dot{X}^\mu \dot{X}^\nu + X'^\mu X'^\nu) = 0, \quad (3.3)$$

$$G_{\mu\nu}\dot{X}^\mu X'^\nu = 0. \quad (3.4)$$

For a diagonal target metric, as in the present case, the energy and the angular momentum in a generic angle φ are given by

$$E = -\frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma G_{00}(X) \partial_\tau X^0, \quad J_\varphi = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma G_{\varphi\varphi}(X) \partial_\tau \varphi. \quad (3.5)$$

Combining these equations, below we find $E = E(J_{\varphi_i})$ for different configurations.

3.1. Glueball states and the Regge string

In comparing masses of glueballs to masses of Kaluza-Klein states, in this section we will only consider classical string states of high spin. Supergravity glueballs will be discussed in section 5. We stress that, in general, the calculation of masses at $\lambda \gg 1$ based on the geometry (2.1) cannot be extrapolated to the weak coupling regime $\lambda \ll 1$ (see section 2 and also discussion in section 4).

To look for classical string solutions, it is convenient to use the radial coordinate ρ which is regular at the horizon located at $\rho = 0$ (in working in the singular u coordinate frame one misses some solutions of strings lying on the horizon).

To describe a string rotating in \mathfrak{R}^3 , we use cylindrical coordinates $dx_1^2 + dx_2^2 + dx_3^2 = dr^2 + r^2 d\beta^2 + dx_3^2$, and consider the following configuration:

$$t = \kappa \tau, \quad \rho = 0, \quad r = r(\sigma), \quad \beta = \omega \tau. \quad (3.6)$$

It represents a folded closed string located at the horizon which rotates in \mathfrak{R}^3 . The action becomes

$$I = -\frac{1}{4\pi} \frac{8\pi\lambda}{3} u_0^2 \int d\tau d\sigma (\kappa^2 - r^2 \omega^2 + r'^2). \quad (3.7)$$

The equations of motion are consistent with those of the original action. The equation of motion for the variable r determines $r = r_0 \sin(\omega\sigma)$. The periodic boundary condition for r then implies that ω must be an integer. We shall consider $\omega = 1$, which corresponds to the minimal one-winding case with the two foldings of the string located at $r = r_0$ (the string rotates around its center of mass at $r = 0$). The Virasoro constraint (3.3) imposes $\kappa^2 = r_0^2$. Then, using (3.5), we find that the energy E and the angular momentum J_β are related by the Regge relation

$$E_{\text{regge}} = 4u_0 \sqrt{\frac{\pi}{3} \lambda J_\beta}, \quad (3.8)$$

corresponding to the string tension (2.4). This relation was pointed out in this context in [19]. For a string with a generic integer $\omega \neq 1$, the energy formula (3.8) is multiplied by a factor $\sqrt{\omega}$.

In order to compare the different energy formulas, in what follows it will be useful to consider the ratio with the string tension, which is the natural scale in pure QCD. From (2.4) and (3.8) we obtain

$$\frac{E_{\text{regge}}}{\sqrt{T_{\text{YM}}}} = 2\sqrt{\pi J_\beta} . \quad (3.9)$$

The above example of the Regge string illustrates an important feature of the string spectrum. Namely, all glueball states corresponding to strings lying at $\rho = 0$ have the same behavior $E_{\text{glueball}}/\sqrt{T_{\text{YM}}} = \text{const.}$ The reason is that once we set $\rho = 0$ the string configurations on \mathbb{R}^3 are the same as in flat space upon changing the flat space string tension T by T_{YM} (2.4). Since this is the only dimensionful scale and the parameter λ does not appear independently, the energies for all these strings must be proportional to $\sqrt{T_{\text{YM}}}$. This means that in units of T_{YM} , to leading order in the expansion in $1/\lambda$, the energies of strings lying at $\rho = 0$ do not change as λ is decreased. This is in contrast to some strings with Kaluza-Klein quantum numbers examined below, and in contrast with supergravity glueballs (see section 5), which have a $1/\sqrt{\lambda}$ leading behavior.

The strings lying at $\rho = 0$ cover a significant part of the glueball string spectrum; besides the special Regge string pointed out above one can write the full flat space spectrum of four dimensional strings. The glueballs that are not included are those where the string also fluctuate in the radial direction ρ . This case is examined in section 5, and the energy exhibits a different dependence on λ .

3.2. String spinning in the cigar

We start with the Nambu-Goto action and consider the ansatz.

$$t = \tau , \quad u = u(\sigma) , \quad \theta_1 = \omega_1 \tau . \quad (3.10)$$

This represents a folded string spinning in the “cigar” geometry described by ρ, θ_1 (or u, θ_1). This configuration first appeared in [19]. Here we are interested in the dependence of the energy as a function of λ . The configuration has a non-trivial quantum number, the angular momentum J_{θ_1} in the internal coordinate θ_1 , and therefore is of Kaluza-Klein type.

The string has energy and angular momentum given by

$$E_{\text{KK}} = \frac{8\lambda}{3} u_0 n \sqrt{y_m^3 - 1} \int_1^{y_m} dy \frac{y^3}{\sqrt{(y_m^3 - y^3)(y^3 - 1)}} , \quad y_m^3 \equiv \frac{\omega_1^2}{\omega_1^2 - 9u_0^2} , \quad (3.11)$$

$$J_{\theta_1} = \frac{8\lambda}{9} n y_m^{3/2} \int_1^{y_m} dy \frac{\sqrt{y^3 - 1}}{\sqrt{y_m^3 - y^3}} . \quad (3.12)$$

Here n is an integer denoting the number of foldings (windings) of the string. The string is extended from $u = u_0$ to $u = u_{\text{max}} = u_0 \sqrt{y_m}$ (the center of mass being at $u = u_0$). If

$\omega_1 \rightarrow 3u_0$, then $u_{\max} \rightarrow \infty$. These equations define $E_{\text{KK}} = E_{\text{KK}}(J_{\theta_1}, n, \lambda)$ in a parametric way. The numeric plot shows that $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}}$ is a monotonically decreasing function of λ . This seems to be a universal behavior that will hold for all stable configurations examined below.

3.3. Strings on $\mathbb{R}^3 \times S^4$

i) The simplest state with S^4 charge is given by a point-like string rotating at the equator of S^4 . It is described by

$$t = \kappa \tau, \quad \alpha = 0, \quad \theta = \frac{\pi}{2} \quad \text{and} \quad \varphi = \omega t. \quad (3.13)$$

Now we find that the energy is given by

$$E_{\text{KK}} = 2u_0 J_\varphi, \quad (3.14)$$

so that

$$\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}} = \frac{1}{\sqrt{\lambda}} \sqrt{3} J_\varphi. \quad (3.15)$$

Comparing the λ dependence of the energies (3.14) and (3.8), corresponding to the Regge string rotating \mathbb{R}^3 , we see that the latter has an extra factor of $\sqrt{\lambda}$. As mentioned above, as λ is decreased (while keeping a large value $\lambda \gg 1$ so that the supergravity approximation still applies) we see that $\frac{E_{\text{regge}}}{\sqrt{T_{\text{YM}}}}$ remains constant and $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}}$ increases (in this case like $\frac{1}{\sqrt{\lambda}}$).

ii) One can also write down a solution describing strings spinning on S^4 . Now we use the Nambu-Goto action and consider the ansatz

$$t = \tau, \quad \rho = 0, \quad \alpha = 0, \quad \theta = \theta(\sigma), \quad \varphi = \omega \tau.$$

This is essentially the same solution appearing in $AdS_5 \times S^5$ [17], given in terms of elliptic functions. The equation for θ is that of a pendulum in a constant gravitational field. There are folded solutions (representing a string extended in the θ direction from $\theta = 0$ to some maximum value $< \pi/2$) and circular solutions (strings winding around the θ direction of S^4). For folded strings, there are two limits, short and long strings. In the case of long strings, one obtains, similarly as in [17], $E_{\text{KK}} \cong 2u_0 J_\varphi$. For short strings, one obtains $E_{\text{KK}} \gg u_0 J_\varphi$. For fixed J_φ , varying λ from 0 to ∞ interpolates between the long and short string regime. One can show numerically that the derivative of the ratio $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}}$ with respect to λ is negative for all λ .

In the case of strings winding in the θ direction we find

$$E_{\text{KK}} = \frac{4\lambda}{3} n u_0 \int_0^{2\pi} d\theta \frac{1}{\sqrt{1 - \eta^2 \sin^2 \theta}}, \quad J_\varphi = \frac{2\lambda}{3} n \eta \int_0^{2\pi} d\theta \frac{\sin^2 \theta}{\sqrt{1 - \eta^2 \sin^2 \theta}}, \quad (3.16)$$

where $\eta = \frac{\omega}{2u_0} < 1$ and n is the winding number, $\theta(\sigma + 2\pi) = \theta(\sigma) + 2\pi n$. These equations define $E_{KK}(J_\varphi, n, \lambda)$. The equations can also be written as $E = E(J_\varphi, n, \eta)$, $\lambda = \lambda(J_\varphi, n, \eta)$. Fixing J_φ and n one can plot the energy as a function of λ . A plot of $\frac{E_{KK}}{\sqrt{T_{YM}}}$ as a function of λ shows that for λ higher than some critical value the derivative of $\frac{E_{KK}}{\sqrt{T_{YM}}}$ becomes positive. The transition occurs at a point $\eta_1 \cong 0.909$, with η_1 being the root of the equation $K(\eta^2) = 2E(\eta^2)$, K and E being the standard elliptic functions. The critical λ is then $\lambda_1 = \lambda(J_\varphi, n, \eta_1)$. The derivative of $\frac{E_{KK}}{\sqrt{T_{YM}}}$ is positive for $\eta < \eta_1$, which corresponds to $\lambda > \lambda_1$.

On the other hand, the string becomes unstable for an angular momentum below a certain critical value (for fixed J_φ , the string is unstable for λ higher than some critical value). This instability is easy to understand in the extreme limit $\omega = 0$; one has a winding string with no angular momentum, and a small perturbation makes the string to slide off the equatorial plane of the sphere due to the string tension. To determine the critical value, one must look at the Lagrangian of small fluctuations. Expanding $\alpha = 0 + \tilde{\alpha}(\theta, t)$ we find

$$L_{\text{fluc}} \cong \frac{1}{u_0} \frac{1}{\sqrt{1 - \eta^2 \sin^2 \theta}} \dot{\tilde{\alpha}}^2 + 4u_0 \frac{1 - 2\eta^2 \sin^2 \theta}{\sqrt{1 - \eta^2 \sin^2 \theta}} \tilde{\alpha}^2 - 4u_0 \sqrt{1 - \eta^2 \sin^2 \theta} \tilde{\alpha}'^2.$$

In the case $\omega = 0$, i.e. $\eta = 0$, the instability is evident since the potential is negative, and one gets harmonic modes with imaginary frequencies. For $\eta \neq 0$, an exact study of the stability requires solving a coupled system of an infinite set of harmonic oscillators. Nevertheless, one can probe the system by a small perturbation at $\tau = 0$ of the form $\tilde{\alpha}(\theta, 0) = \delta_0$. Then one sees immediately that the potential becomes negative precisely at $\eta < \eta_1 \cong 0.909$, and the equation for the zero mode at small times is that of a harmonic oscillator with imaginary frequency, indicating instability. Thus instabilities seem to appear precisely at the same point where the derivative of $\frac{E_{KK}}{\sqrt{T_{YM}}}$ with respect to λ becomes positive. This remarkable feature is seen more clearly in other solutions examined below. Namely, the solutions become unstable precisely in the same interval where the derivative of the ratio $\frac{E_{KK}}{\sqrt{T_{YM}}}$ with respect to λ is positive. The instability is of the same type, there is a winding string that slides off a big circle of the sphere when the angular momentum is lower than some critical value.

iii) The string spins on the $\mathbb{R}^3 \times S^4$ space

$$\rho = 0, \quad t = \kappa \tau, \quad r = r_0, \quad \beta = m(\sigma + \tau), \quad \alpha = 0, \quad \theta = \frac{\pi}{2}, \quad \varphi = -n\sigma + \omega\tau. \quad (3.17)$$

The Virasoro constraint (3.4) implies the relation $nJ_\varphi = mJ_\beta$ between the angular momenta associated with the angles φ and β . Now we find

$$4u_0^2 r_0^2 m^2 = n\omega, \quad J_\varphi = \frac{2\pi\lambda}{3} \omega,$$

$$E_{\text{KK}} = 2 u_0 J_\varphi + \frac{4\pi\lambda}{3} u_0 n, \quad (3.18)$$

i.e.

$$\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}} = \frac{\sqrt{3}}{\sqrt{\lambda}} J_\varphi + \frac{2\pi\sqrt{\lambda}}{\sqrt{3}} n. \quad (3.19)$$

As λ is decreased, the ratio $E_{\text{KK}}/\sqrt{T_{\text{YM}}}$ (3.19) of these states first decreases and then increases. The transition is at $\lambda_0 = \frac{3J_\varphi}{2\pi n}$. For states with $\lambda_0 = \frac{3J_\varphi}{2\pi n} \gg 1$, the energy formula applies in the region $\lambda < \lambda_0$, so they exhibit the behavior $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}} \sim \frac{1}{\sqrt{\lambda}}$. On the other hand, for states with $\frac{3J_\varphi}{2\pi n} \ll 1$, the formula (3.19) cannot be extrapolated into the region $\lambda < \lambda_0$.

Remarkably, these string configurations become classically unstable precisely when $J_\varphi < \frac{2\pi\lambda}{3} n$, i.e. precisely when the derivative of $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}}$ with respect to λ is positive. This can be seen by examining quadratic fluctuations in the θ direction, $\theta = \frac{\pi}{2} + \tilde{\theta}(\tau)$. We get the Lagrangian

$$L_{\text{fluc}} \cong \dot{\tilde{\theta}}^2 - (\omega^2 - n^2)\tilde{\theta}^2,$$

so there is an instability mode for $\omega < n$. Since $J_\varphi = \frac{2\pi\lambda}{3} \omega$, this corresponds to $J_\varphi < \frac{2\pi\lambda}{3} n$, i.e. the string is unstable when $\lambda > \lambda_0$. The instability mode can be understood as a tendency of the circular string wrapping the S^4 to slip off the side when the angular momentum is insufficient to keep it stable.

Thus, for all stable string configurations of this type $E_{\text{KK}}/\sqrt{T_{\text{YM}}}$ increases in decreasing λ . In particular, for states with $J_\varphi \gg n$, the increasing regime already starts at $\lambda \gg 1$, so their energy reproduces the $\frac{1}{\sqrt{\lambda}}$ behavior of previous cases.

3.4. String on $\mathbb{R}^3 \times S^1$

The configuration

$$t = \kappa \tau, \quad u = u_1, \quad \theta_1 = n_1 \sigma - \omega_1 \tau, \quad r = r_0, \quad \beta = m(\sigma + \tau), \quad (3.20)$$

with $\omega, n, m > 0$, solves the equations of motion and Virasoro constraints with $(\frac{u_1}{u_0})^6 = \frac{\omega_1}{n_1}$ and with the relation between the parameters $\omega_1 - n_1 = (3mu_0r_0)^2/n_1$ (n_1 and m are integers). The energy of the configuration is

$$E_{\text{KK}} = \frac{8\pi\lambda}{9} u_0 n_1 \left(\frac{\omega_1}{n_1} + 1 \right) \sqrt{\frac{\omega_1}{n_1} - 1}, \quad (3.21)$$

whereas the (integer) momentum p_{θ_1} over S^1 is given by

$$p_{\theta_1} = \frac{8\pi\lambda}{27} n_1 \left(\frac{\omega_1}{n_1} - 1 \right) \sqrt{\frac{\omega_1}{n_1}}. \quad (3.22)$$

From the constraint (3.4) one has the relation $J_\beta m = p_{\theta_1} n_1$.

The λ dependence of the energy in (3.21) is hidden in $\omega_1 = \omega_1(\lambda)$, defined in terms of the quantized momentum $p_{\theta_1} = 0, \pm 1, \pm 2, \dots$ and λ through (3.22). One can write explicitly $\omega_1 = \omega_1(\lambda)$ by solving the cubic equation (3.22). The function $\omega_1(\lambda)/n_1$ tends to infinity at $\lambda = 0$ and it approaches 1 at $\lambda = \infty$. The energy $\frac{E_{\text{KK}}(\lambda)}{\sqrt{T_{\text{YM}}}}$ goes like $\sim \frac{1}{\sqrt{\lambda}}$ near $\lambda = 0$ and approaches a constant at $\lambda = \infty$. This can be seen explicitly by considering limits. In the case of large λ , $\frac{\omega_1}{n_1} - 1 \ll 1$ we find²

$$E_{\text{KK}} \cong 4\sqrt{\frac{2\pi\lambda}{3}}\sqrt{p_{\theta_1}n_1}u_0, \quad (3.23)$$

For $\lambda \ll p_{\theta_1}/n_1$ one has $\frac{\omega_1}{n_1} \gg 1$ and we obtain $E \cong 3u_0p_{\theta_1}$, showing that $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}} \sim \frac{1}{\sqrt{\lambda}}$ for $1 \ll \lambda \ll p_{\theta_1}/n_1$.

3.5. String spinning on $S^1 \times S^4$

i) We now consider the configuration:

$$t = \kappa\tau, \quad u = u_1, \quad \theta_1 = \omega_1\tau - n_1\sigma, \quad \alpha = 0, \quad \theta = \frac{\pi}{2}, \quad \varphi = m(\sigma + \tau), \quad (3.24)$$

with $w_1, n_1, m > 0$. For this configuration

$$\left(\frac{u_1}{u_0}\right)^6 = \frac{\omega_1}{n_1}, \quad \kappa^2 = \frac{(\omega_1 - n_1)(\omega_1 + n_1)^2}{9u_0^2\omega_1}.$$

Note that the configuration only exists for $\omega_1 > n_1$, i.e. for $u_1 > u_0$. After some straightforward manipulations, one gets the following formulas for the energy and the momenta:

$$E_{\text{KK}} = \frac{8\pi\lambda}{9}u_0\sqrt{\frac{\omega_1}{n_1} - 1}(\omega_1 + n_1), \quad (3.25)$$

$$p_{\theta_1} = \frac{8\pi}{27}\lambda(\omega_1 - n_1)\sqrt{\frac{\omega_1}{n_1}}, \quad J_\varphi = \frac{4\pi}{9}\lambda\left(\frac{\omega_1}{n_1}\right)^{\frac{1}{3}}\sqrt{n_1(\omega_1 - n_1)}, \quad (3.26)$$

$$m = \frac{2}{3}\left(\frac{\omega_1}{n_1}\right)^{\frac{1}{6}}\sqrt{n_1(\omega_1 - n_1)}. \quad (3.27)$$

Note that $J_\varphi m = p_{\theta_1} n_1$. These relations are similar to the ones appearing for the configuration of sect. 3.4, except that the solution of section 3.4 has the additional parameter r_0 .

² Note that, since $n_1 \gg \omega_1 - n_1 \cong \frac{27}{8\pi}\frac{p_{\theta_1}}{\lambda}$, it follows that $\lambda \gg p_{\theta_1}/n_1$ and therefore $E_{\text{KK}} \gg u_0p_{\theta_1}$. Since p_{θ_1} is quantized, this implies $E_{\text{KK}} \gg u_0$.

This string solution has the remarkable property that it exists only for a special value of λ . Indeed, fixing four integer quantum numbers satisfying $J_\varphi m = p_{\theta_1} n_1$, then the two equations (3.26) determine ω_1 and λ in terms of integer numbers. The equation (3.27) is then satisfied automatically. For a small variation of λ , this string state ceases to be physical and disappears from the string spectrum. In other words, there is a set of special values of λ (parametrized by integers $J_\varphi, p_{\theta_1}, n_1$) for which the string spectrum has extra states.

Explicit formulas for the energy of the state in terms of the quantum numbers can be obtained in different limits. In the limit $\frac{\omega_1}{n_1} - 1 \ll 1$, the energy becomes

$$E_{\text{KK}} \cong 4 u_0 J_\varphi. \quad (3.28)$$

In the opposite limit $\frac{\omega_1}{n_1} \gg 1$, we obtain

$$E_{\text{KK}} \cong 3 u_0 p_{\theta_1}. \quad (3.29)$$

ii) Here we use the alternative parametrization of S^4 ,

$$d\Omega_4^2 = d\gamma^2 + \cos^2 \gamma d\varphi_1^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\varphi_2^2). \quad (3.30)$$

Now the configuration considered is

$$t = \kappa\tau, \quad u = u_1, \quad \theta_1 = \omega_1\tau, \quad \gamma = m\sigma, \quad \psi = 0, \quad \varphi_2 = \omega\tau, \quad \varphi_1 = \omega\tau. \quad (3.31)$$

This configuration appeared in [20]. From the equations of motion and the Virasoro constraints we get

$$\left(\frac{u_1}{u_0}\right)^4 = \frac{4\omega_1^2}{6m^2 + 3\omega^2}, \quad \kappa^2 = \frac{8\omega_1^3 + 3\sqrt{3}(m^2 + 2\omega^2)\sqrt{2m^2 + \omega^2}}{72u_0^2\omega_1}. \quad (3.32)$$

The particular case considered in [20], section 6, corresponds to $u_1 = u_0$, which implies $4\omega_1^2 = 6m^2 + 3\omega^2$ and then $\kappa^2 = \frac{m^2 + \omega^2}{4u_0^2}$. The horizon limit $u_1 \rightarrow u_0$ ($\rho \rightarrow 0$) is smooth for this configuration.

The energy, the angular momentum $J_\varphi := J_{\varphi_1} = J_{\varphi_2}$ and the momentum p_{θ_1} are, respectively

$$E_{\text{KK}} = \frac{8\pi}{3} \lambda \frac{u_1^3}{u_0} \kappa, \quad J_\varphi = \frac{\pi}{3} \lambda \omega \frac{u_1}{u_0}, \quad p_{\theta_1} = \frac{8\pi}{27} \lambda \left(\frac{u_1}{u_0}\right)^3 \left(1 - \left(\frac{u_0}{u_1}\right)^6\right) \omega_1. \quad (3.33)$$

For a string lying near the horizon, $u_1 \sim u_0$, the energy becomes

$$E_{\text{KK}} \cong 4u_0 \sqrt{J_\varphi^2 + \left(\frac{\pi}{3} \lambda m\right)^2}. \quad (3.34)$$

Hence

$$\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}} \cong 2\sqrt{3}\sqrt{\frac{J_\varphi^2}{\lambda} + \left(\frac{\pi}{3}m\right)^2\lambda}. \quad (3.35)$$

Similarly to the discussion in section 3.2, we note that, as λ is decreased, the ratio $E_{\text{KK}}/\sqrt{T_{\text{YM}}}$ of these states first decreases and then increases. The transition is now at $\lambda_0 = \frac{3}{\pi m}J_\varphi$. For states with $\lambda_0 = \frac{3}{\pi m}J_\varphi \gg 1$, the energy formula applies in the region $\lambda < \lambda_0$, so they exhibit the behavior $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}} \sim \frac{1}{\sqrt{\lambda}}$ of the previous KK configurations. On the other hand, for states with $\frac{3}{\pi m}J_\varphi \ll 1$, (3.35) cannot be extrapolated into the region $\lambda < \lambda_0$.

Just as the strings ii) and iii) of section 3.3, these string configurations become classically unstable precisely when $J_\varphi < \frac{\pi}{3}\lambda m$, i.e. when the derivative of $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}}$ with respect to λ is positive. To identify instability modes it is convenient to examine the configuration in terms of the coordinates (2.2),

$$\alpha = 0, \quad \theta = \frac{\pi}{4}, \quad \phi = \omega\tau + m\sigma, \quad \varphi = \omega\tau - m\sigma. \quad (3.36)$$

Then the instability shows up by examining quadratic fluctuations in the α direction. We get the Lagrangian

$$L_{\text{fluc}} \cong \dot{\tilde{\alpha}}^2 - (\omega^2 - n^2)\tilde{\alpha}^2,$$

exhibiting an instability mode precisely for $\omega < n$, i.e. for $J_\varphi < \frac{\pi}{3}\lambda m$.

Now consider $u_1/u_0 \gg 1$. In this case the energy comes mostly from the motion in θ_1 and from (3.32), (3.33) we get

$$E_{\text{KK}} \cong 3u_0 p_{\theta_1}. \quad (3.37)$$

Thus for $u_1/u_0 \gg 1$ one has the behavior $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}} \sim \frac{1}{\sqrt{\lambda}}$.

There are also circular string configurations similar to those studied in [23]. In the present case the circular string lies on the horizon at $\rho = 0$, and the solutions are essentially those of [23]. The energy has a similar structure as (3.34) and in the regime $\lambda \ll J/m$, where the angular momentum term dominates, it has the behavior $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}} \sim \frac{1}{\sqrt{\lambda}}$.

4. A small slice of the $\lambda \ll 1$ spectrum

As pointed out in section 2, the scalar curvature of the geometry (2.1) is small in units of α' provided

$$\frac{u_0}{\lambda u} \ll 1. \quad (4.1)$$

One can check that this condition ensures that all curvature invariants are small. It also ensures that higher derivatives of the dilaton field are small. By considering strings moving

on the region $u \gg u_0/\lambda$, one can therefore describe the geometry in terms of the leading order metric (2.1), since in this region higher order α' corrections can be neglected. For such strings, corrections to the classical energy formula could therefore only originate from quantum sigma model corrections, but not from corrections to the background. However, assuming as usual that the classical limit exists, quantum corrections to the energy can also be neglected relative to the leading term by taking large quantum numbers. This should be true even for small λ , provided one takes sufficiently large quantum excitation numbers, such as the angular momentum. For small λ , the supergravity background will receive important corrections, but not in the region $u \gg u_0/\lambda$.

Among the different solutions of the previous sections, there are several ones which can entirely lie in the $u \gg u_0/\lambda$ region. These are the configurations: 1) (3.20), 2) (3.24) and 3) (3.31). Increasing the frequency ω_1 around θ_1 produces the effect that the equilibrium position u_1 of the string moves towards larger values of u . In the cases 1) and 2), demanding that $u_1 \gg u_0/\lambda$ requires $\omega_1 \gg n_1/\lambda^6$. For $\lambda < 1$, this implies that $p_{\theta_1} \gg n_1/\lambda^8$. In this limit, the energy formulas in both cases 1) and 2) are given by

$$1), 2) : \quad E_{\text{KK}} \cong 3u_0 p_{\theta_1} \gg \frac{u_0 n_1}{\lambda^8} . \quad (4.2)$$

We recall that the solution 2) (3.24) exists only for a discrete set of λ parametrized by the quantum numbers $J_\varphi, p_{\theta_1}, n_1$. For the case 3) (3.31), the string lies in the $u_1 \gg u_0/\lambda$ region provided $\omega_1 \gg \sqrt{2m^2 + \omega^2}/\lambda^2$, i.e. $p_{\theta_1} \gg \sqrt{2m^2 + \omega^2}/\lambda^4$. This implies $p_{\theta_1} \gg m/\lambda^4$. In this limit, the energy formula becomes

$$3) \quad E_{\text{KK}} \cong 3u_0 p_{\theta_1} \gg \frac{u_0 m}{\lambda^4} . \quad (4.3)$$

For the reasons explained above, the energy formulas (4.2) and (4.3) should apply for small values of λ provided p_{θ_1} is sufficiently large as prescribed (to guarantee that the string lies in a small curvature region) and large angular momentum (to suppress quantum sigma model corrections relative to the leading contribution to the energy).

These formulas indicate a rapid decoupling of these Kaluza-Klein states.

5. Description of supergravity glueballs from classical geodesic motion

Masses of glueballs of the supergravity sector can be obtained by solving the equations of motion corresponding to supergravity fluctuations [2-9]. In particular, the spectrum of a scalar glueball 0^{++} is obtained from the supergravity equation for the dilaton mode fluctuation,

$$\partial_\mu (\sqrt{g} e^{-2\phi} g^{\mu\nu} \partial_\nu \tilde{\phi}) = 0 . \quad (5.1)$$

Setting

$$\tilde{\phi} = e^{ik \cdot x} \chi(u), \quad (5.2)$$

one obtains

$$\frac{1}{u^3} \partial_u [u(u^6 - u_0^6) \partial_u \chi(u)] + M^2 \chi(u) = 0, \quad M^2 = -k^2. \quad (5.3)$$

Using the WKB method, for large masses one can approximate the spectrum by [5]

$$M \cong \frac{\pi}{\xi} \sqrt{n(n+2)} u_0 \cong \frac{\pi}{\xi} n u_0, \quad (5.4)$$

with n integer and

$$\xi = u_0 \int_{u_0}^{\infty} du \frac{1}{u^2 \sqrt{h(u)}} = \frac{\sqrt{\pi} \Gamma(\frac{7}{6})}{\Gamma(\frac{2}{3})}, \quad h(u) = 1 - \frac{u_0^6}{u^6}. \quad (5.5)$$

Now we will show that this glueball spectrum can be alternatively obtained by considering a point-like string configuration oscillating along the variable u between the horizon and infinity. To be specific, we consider a configuration: $u = u(\tau)$, $t = t(\tau)$, describing a point-like string which oscillates along a meridian $\theta_1 = \theta_0$ of the “cigar” described by the coordinates u and θ_1 , from $u = \infty$ to $u = u_0$ and then back to $u = \infty$ following the “other side” of the meridian, $\theta_1 = \theta_0 + \pi$. In the conformal gauge the equations of motion set

$$\dot{t} = \frac{c}{u^3}, \quad \dot{u} = \pm \frac{c}{u} \sqrt{h(u)},$$

which describe a null geodesic in the background (2.1). The plus sign corresponds to the branch describing the string coming from ∞ to $u = u_0$, whereas the minus sign corresponds to the branch describing the string coming from $u = u_0$ to ∞ . The (target-space) time elapsed in a motion from $u = \infty$ to u_0 and back to $u = \infty$ is

$$\Delta T = \int dt = 2 \int_{u_0}^{\infty} \frac{\dot{t}}{\dot{u}} du = 2 \int_{u_0}^{\infty} du \frac{1}{u^2 \sqrt{h(u)}} = \frac{2\sqrt{\pi} \Gamma(\frac{7}{6})}{u_0 \Gamma(\frac{2}{3})} = \frac{2\xi}{u_0}, \quad (5.6)$$

which is finite, so the motion is periodical. One can define the “angular” frequency $\omega_0 := \frac{2\pi}{\Delta T} = \frac{\pi u_0}{\xi}$.

In order to determine the quantum energy levels, it is more convenient to work with the Nambu-Goto action. We will consider a more general string configuration $u = u(\tau)$, $\theta_1 = m\sigma$, and set the gauge $\tau = t$. The case $m = 0$ will be recovered at the end. With this configuration we get the Lagrangian

$$\mathcal{L} = -\frac{4}{9} m \lambda u_0 \sqrt{\frac{u^6}{u_0^6} - 1 - \frac{\dot{u}^2 u^2}{u_0^6}}. \quad (5.7)$$

Following [24], we perform a change of variables in order to obtain a Hamiltonian in a form suitable for the application of the WKB approximation. We introduce a new variable ζ

$$\frac{d\zeta}{du} = \frac{u}{\sqrt{u^6 - u_0^6}}.$$

This defines $u = u(\zeta)$. The Lagrangian (5.7) takes the form

$$\mathcal{L} = -\frac{4}{9}m\lambda u_0 \sqrt{\left(\frac{u(\zeta)^6}{u_0^6} - 1\right)(1 - \dot{\zeta}^2)}. \quad (5.8)$$

The Hamiltonian is then given by

$$H = \sqrt{p_\zeta^2 + \left(\frac{4}{9}m\lambda u_0\right)^2 \left(\frac{u(\zeta)^6}{u_0^6} - 1\right)}. \quad (5.9)$$

Note that $\frac{1}{2}H^2$ describes a one dimensional system with potential $V = \frac{1}{2}(\frac{4}{9}m\lambda)^2(\frac{u^6}{u_0^6} - 1)u_0^2$. The potential is zero at the horizon $u = u_0$ and diverges at $u = \infty$. We can consider $u - u_0$ taking positive and negative values as well, with a symmetric potential and taking into account only even wave functions. In this case the Bohr-Sommerfeld formula gives the quantization rule ($n \gg 1$)

$$(2n + \frac{1}{2})\pi \cong \int d\zeta \sqrt{E^2 - \left(\frac{4}{9}m\lambda u_0\right)^2 \left(\frac{u(\zeta)^6}{u_0^6} - 1\right)} = 2 \int_{u_0}^{u_1} du u \sqrt{\frac{E^2}{u^6 - u_0^6} - \left(\frac{4}{9} \frac{m\lambda}{u_0^2}\right)^2},$$

with

$$u_1 = \frac{u_0}{\sqrt{\eta}}, \quad \eta \equiv \frac{1}{\left(1 + \left(\frac{9E}{4m\lambda u_0}\right)^2\right)^{1/3}}.$$

For $m = 0$ we obtain

$$E \cong \left(n + \frac{1}{4}\right)\omega_0 \cong n\omega_0 = \frac{\pi}{\xi} n u_0.$$

This exactly agrees with (5.4). Thus one can think of supergravity glueballs as point-like strings oscillating in the radial direction. Note that the classical configuration does not distinguish parity, i.e. it equally describes 0^{++} or 0^{-+} glueballs. In [5,9], the difference in their energies is seen to be subleading in radial number n , so as expected in the classical limit $n \rightarrow \infty$ their energies are the same.

In order to obtain the energy formula for $m \neq 0$, we write

$$(2n + \frac{1}{2})\pi \cong \frac{E}{u_0} \sqrt{\frac{\eta}{1 - \eta^3}} \int_\eta^1 dy \sqrt{\frac{1 - y^3}{y^3 - \eta^3}}, \quad y = \frac{u^2}{u_1^2}. \quad (5.10)$$

We can expand (5.10) for small η , corresponding to $E \gg m\lambda u_0$. This leads to the expansion

$$(2n + \frac{1}{2})\pi = \frac{E}{u_0} \frac{2\sqrt{\pi}\Gamma(\frac{7}{6})}{\Gamma(\frac{2}{3})} - \frac{2^{\frac{2}{3}}\sqrt{\pi}\Gamma(\frac{5}{6})}{3^{\frac{2}{3}}\Gamma(\frac{4}{3})} \left(\frac{E}{u_0}\right)^{\frac{2}{3}} m^{\frac{1}{3}} \lambda^{\frac{1}{3}} + \dots \quad (5.11)$$

i.e.

$$E \cong u_0 (2.59 n + 1.33 m^{\frac{1}{3}} \lambda^{\frac{1}{3}} n^{\frac{2}{3}} + \dots) . \quad (5.12)$$

A different limit is $\lambda \gg 1$ so that $\eta \sim 1$. In this case one can check that the energy asymptotically approaches the behavior $E \sim \text{const.} \sqrt{\lambda} u_0$. The derivative of the ratio $\frac{E}{\sqrt{T_{\text{YM}}}}$ with respect to λ is negative for all λ .

6. Concluding remarks

We have determined the energy formula for different string configurations in a string model dual to pure $SU(N)$ Yang-Mills in 3+1 dimensions. The dependence of the energy on λ can be physically understood as follows. The energy of a rotating string which is wound in some direction has typically a kinetic contribution proportional to the angular momentum, and a “rest mass” contribution proportional to the string tension times the winding number times the length. For angular momentum in one of the S^4 directions, the kinetic contribution is typically independent of λ and this is what produces the behavior $E \sim u_0$ or, equivalently, $\frac{E}{\sqrt{T_{\text{YM}}}} \sim \frac{1}{\sqrt{\lambda}}$ at λ small enough so that the “rest mass” contribution is negligible (λ can still be $\lambda \gg 1$, depending on the quantum numbers). For a string wound around S^4 , the string tension T_{S^4} is proportional to λ (just as the string tension T_{YM} of the strings in \mathbb{R}^3). As a result, the ratio $\frac{E}{\sqrt{T_{\text{YM}}}}$ is of order $\sqrt{\lambda}$ at large λ . This is the reason of the large λ behavior of (3.19) and (3.35).

For low angular momentum, or for sufficiently large string tension, the circular strings winding the sphere become unstable. This is expected, since the string can slide off the big circle of the sphere. But what is remarkable is that the instability precisely appears at the same point where the derivative of $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}}$ with respect to λ changes sign. This phenomenon arose in three independent cases. Consequently, for all classically stable strings with non-trivial Kaluza-Klein quantum numbers examined here, the ratio $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}}$ always increases as λ is decreased. This behavior of the energy also holds for supergravity glueballs. On the other hand, the energy of all glueballs associated with strings (with trivial Kaluza-Klein quantum numbers) lying on the horizon exhibits the behavior $\frac{E}{\sqrt{T_{\text{YM}}}} \sim \text{const.}$ In sum, the general pattern that we find is that the energy of all classically stable states considered here does not increase faster than $\sqrt{\lambda}$ at large λ . It would be interesting to see if this is a property of all states in the string spectrum.

The fact that for all Kaluza-Klein states the ratio $\frac{E_{\text{KK}}}{\sqrt{T_{\text{YM}}}}$ increases as λ is decreased, while for all 3+1 strings lying at the horizon this ratio remains constant, could be viewed

as an indication of the presence of two distinct scales which may separate as λ is gradually decreased. However, one should keep in mind that in general these results cannot be extrapolated to $\lambda \ll 1$ (except for states living sufficiently far from the horizon as those of section 4). Also, the energy of supergravity glueballs exhibits the same behavior of KK states, i.e. $\frac{E}{\sqrt{T_{\text{YM}}}}$ increases as λ is decreased, and they are not expected to decouple, since the dilaton fluctuation is dual to the operator $\text{Tr } F^2$.

In section 5, we have discussed how to describe supergravity glueballs in terms classical configurations oscillating in the radial direction. It would be interesting to similarly describe glueballs of higher spin (> 2) in terms of strings oscillating in the radial direction with rotation in the angle β .

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References

- [1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)] [arXiv:hep-th/9711200].
- [2] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Adv. Theor. Math. Phys.* **2**, 505 (1998) [arXiv:hep-th/9803131].
- [3] C. Csaki, H. Ooguri, Y. Oz and J. Terning, “Glueball mass spectrum from supergravity,” *JHEP* **9901**, 017 (1999) [arXiv:hep-th/9806021].
- [4] H. Ooguri, H. Robins and J. Tannenhauser, “Glueballs and their Kaluza-Klein cousins,” *Phys. Lett. B* **437**, 77 (1998) [arXiv:hep-th/9806171].
- [5] J. A. Minahan, “Glueball mass spectra and other issues for supergravity duals of QCD models,” *JHEP* **9901**, 020 (1999) [arXiv:hep-th/9811156].
- [6] J. G. Russo, “New compactifications of supergravities and large N QCD,” *Nucl. Phys. B* **543**, 183 (1999) [arXiv:hep-th/9808117].
- [7] C. Csaki, Y. Oz, J. Russo and J. Terning, “Large N QCD from rotating branes,” *Phys. Rev. D* **59**, 065012 (1999) [arXiv:hep-th/9810186].
- [8] J. G. Russo and K. Sfetsos, “Rotating D3 branes and QCD in three dimensions,” *Adv. Theor. Math. Phys.* **3**, 131 (1999) [arXiv:hep-th/9901056].
- [9] C. Csaki, J. Russo, K. Sfetsos and J. Terning, “Supergravity models for 3+1 dimensional QCD,” *Phys. Rev. D* **60**, 044001 (1999) [arXiv:hep-th/9902067].
- [10] A. Hashimoto and Y. Oz, “Aspects of QCD dynamics from string theory,” *Nucl. Phys. B* **548**, 167 (1999) [arXiv:hep-th/9809106].
- [11] D. J. Gross and H. Ooguri, “Aspects of large N gauge theory dynamics as seen by string theory,” *Phys. Rev. D* **58**, 106002 (1998) [arXiv:hep-th/9805129].
- [12] M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, “Towards a holographic dual of large-N(c) QCD,” arXiv:hep-th/0311270.
- [13] J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik and I. Kirsch, “Chiral symmetry breaking and pions in non-supersymmetric gauge / gravity duals,” *Phys. Rev. D* **69**, 066007 (2004) [arXiv:hep-th/0306018].
- [14] J. L. F. Barbon, C. Hoyos, D. Mateos and R. C. Myers, “The holographic life of the eta’,” arXiv:hep-th/0404260.
- [15] A. Armoni, “Witten-Veneziano from Green-Schwarz,” *JHEP* **0406**, 019 (2004) [arXiv:hep-th/0404248].
- [16] M. Bando, T. Kugo, A. Sugamoto and S. Terunuma, “Pentaquark baryons in string theory,” arXiv:hep-ph/0405259.
- [17] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” *Nucl. Phys. B* **636**, 99 (2002) [arXiv:hep-th/0204051].

- [18] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, “Stringing spins and spinning strings,” JHEP **0309**, 010 (2003) [arXiv:hep-th/0306139]; S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors,” Phys. Lett. B **570**, 96 (2003) [arXiv:hep-th/0306143]; G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, “Spinning strings in $AdS(5) \times S^5$ and integrable systems,” Nucl. Phys. B **671**, 3 (2003) [arXiv:hep-th/0307191]; M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin, “Large spin limit of $AdS(5) \times S^5$ string theory and low energy expansion of ferromagnetic spin chains,” arXiv:hep-th/0403120; G. Arutyunov and M. Staudacher, “Two-loop commuting charges and the string / gauge duality,” arXiv:hep-th/0403077.
- [19] A. Armoni, J. L. F. Barbon and A. C. Petkou, “Rotating strings in confining AdS/CFT backgrounds,” JHEP **0210**, 069 (2002) [arXiv:hep-th/0209224].
- [20] M. Alishahiha, A. E. Mosaffa and H. Yavartanoo, “Multi-spin string solutions in AdS black hole and confining backgrounds,” arXiv:hep-th/0402007.
- [21] J. M. Pons and P. Talavera, “Semi-classical string solutions for $N = 1$ SYM,” Nucl. Phys. B **665**, 129 (2003) [arXiv:hep-th/0301178]; L. A. Pando Zayas, J. Sonnenschein and D. Vaman, “Regge trajectories revisited in the gauge / string correspondence,” Nucl. Phys. B **682**, 3 (2004) [arXiv:hep-th/0311190]; F. Bigazzi, A. L. Cotrone and L. Martucci, arXiv:hep-th/0403261.
- [22] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, “Wilson loops, confinement, and phase transitions in large N gauge theories from supergravity,” JHEP **9806**, 001 (1998) [arXiv:hep-th/9803263].
- [23] G. Arutyunov, J. Russo and A. A. Tseytlin, “Spinning strings in $AdS(5) \times S^5$: New integrable system relations,” Phys. Rev. D **69**, 086009 (2004) [arXiv:hep-th/0311004].
- [24] J. A. Minahan, “Circular semiclassical string solutions on $AdS(5) \times S^5$,” Nucl. Phys. B **648**, 203 (2003) [arXiv:hep-th/0209047].